

A NEW EXPLICIT FORMULA FOR BERNOULLI AND GENOCCHI NUMBERS IN TERMS OF STIRLING NUMBERS

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ABSTRACT. In the paper, the authors review some explicit formulas and establish a new explicit formula for Bernoulli and Genocchi numbers in terms of Stirling numbers of the second kind.

1. MAIN RESULTS

It is well known that Bernoulli numbers B_n for $n \geq 0$ may be defined by the power series expansion

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{x^{2k}}{(2k)!}, \quad |x| < 2\pi, \quad (1.1)$$

that Euler polynomials $E_n(x)$ are defined by

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (1.2)$$

that Genocchi numbers G_n for $n \in \mathbb{N}$ are given by the generating function

$$\frac{2t}{e^t + 1} = \sum_{n=1}^{\infty} G_n \frac{t^n}{n!}, \quad (1.3)$$

and that Stirling numbers of the second kind which may be generated by

$$\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!}, \quad k \in \mathbb{N} \quad (1.4)$$

and may be computed by

$$S(k, m) = \frac{1}{m!} \sum_{\ell=1}^m (-1)^{m-\ell} \binom{m}{\ell} \ell^k, \quad 1 \leq m \leq k. \quad (1.5)$$

By the way, Stirling number of the second kind $S(n, k)$ may be interpreted combinatorially as the number of ways of partitioning a set of n elements into k nonempty subsets.

Bernoulli numbers B_n for $n \in \{0\} \cup \mathbb{N}$ satisfy

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_{2n+2} \neq 0, \quad B_{2n+3} = 0. \quad (1.6)$$

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For $n \in \mathbb{N}$, Genocchi numbers meet $G_{2n+1} = 0$. The first few Genocchi numbers G_n are listed in Table 1.1. Genocchi numbers G_{2n} may be represented in terms of

TABLE 1.1. The first few Genocchi numbers G_n

n	1	2	4	6	8	10	12	14	16	18
G_n	1	-1	1	-3	17	-155	2073	-38227	929569	-28820618

Bernoulli numbers B_{2n} and Euler polynomials $E_{2n-1}(0)$ as

$$G_{2n} = 2(1 - 2^{2n})B_{2n} = 2nE_{2n-1}(0), \quad n \in \mathbb{N}. \quad (1.7)$$

See [1, p. 49]. As a result, we have

$$G_n = 2(1 - 2^n)B_n, \quad n \in \mathbb{N}. \quad (1.8)$$

The first formula for Bernoulli numbers B_n listed in [2] is

$$B_n = \sum_{k=0}^n \frac{1}{k+1} \sum_{j=0}^k (-1)^j \binom{k}{j} j^n, \quad n \geq 0, \quad (1.9)$$

which is a special case of the general formula [11, (2.5)]. The formula (1.9) is equivalent to

$$B_n = \sum_{k=0}^n (-1)^k \frac{k!}{k+1} S(n, k), \quad n \in \{0\} \cup \mathbb{N}, \quad (1.10)$$

which was listed in [3, p. 536] and [4, p. 560]. Recently, four alternative proofs of the formula (1.10) were provided in [6, 15]. A generalization of the formula (1.10) was supplied in [5]. In all, we may collect at least seven alternative proofs for the formula (1.9) or (1.10) in [2, 4, 6, 11, 12, 15] and closely related references therein.

In [2, p. 48, (11)], it was deduced that

$$B_n = \sum_{j=0}^n (-1)^j \binom{n+1}{j+1} \frac{n!}{(n+j)!} \sum_{k=0}^j (-1)^{j-k} \binom{j}{k} k^{n+j}, \quad n \geq 0, \quad (1.11)$$

which may be rearranged as

$$B_n = \sum_{i=0}^n (-1)^i \frac{\binom{n+1}{i+1}}{\binom{n+i}{i}} S(n+i, i), \quad n \geq 0. \quad (1.12)$$

The formula (1.12) was rediscovered in the preprint [13]. On 21 January 2014, the authors searched out that the formula (1.12) was also derived in [10, p. 59] and [16, p. 140].

In [9, p. 1128, Corollary], among other things, it was found that

$$B_{2k} = \frac{1}{2} - \frac{1}{2k+1} - 2k \sum_{i=1}^{k-1} \frac{A_{2(k-i)}}{2(k-i)+1} \quad (1.13)$$

for $k \in \mathbb{N}$, where A_m is defined by

$$\sum_{m=1}^n m^k = \sum_{m=0}^{k+1} A_m n^m.$$

In [14, Theorem 1.4], among other things, it was presented that

$$B_{2k} = \frac{(-1)^{k-1}k}{2^{2(k-1)}(2^{2k}-1)} \sum_{i=0}^{k-1} \sum_{\ell=0}^{k-i-1} (-1)^{i+\ell} \binom{2k}{\ell} (k-i-\ell)^{2k-1}, \quad k \in \mathbb{N}. \quad (1.14)$$

In [8, Theorem 3.1], it was obtained that

$$B_{2k} = 1 + \sum_{m=1}^{2k-1} \frac{S(2k+1, m+1)S(2k, 2k-m)}{\binom{2k}{m}} - \frac{2k}{2k+1} \sum_{m=1}^{2k} \frac{S(2k, m)S(2k+1, 2k-m+1)}{\binom{2k}{m-1}}, \quad k \in \mathbb{N}. \quad (1.15)$$

The aim of this paper is to find the following new explicit formula for Bernoulli numbers B_k , or say, Genocchi numbers G_k , in terms of Stirling numbers of the second kind $S(k, m)$.

Theorem 1.1. *For all $k \in \mathbb{N}$, Genocchi numbers G_k may be computed by*

$$G_k = 2(1-2^k)B_k = (-1)^k k \sum_{m=1}^k (-1)^m \frac{(m-1)!}{2^{m-1}} S(k, m). \quad (1.16)$$

2. PROOF OF THEOREM 1.1

Differentiating on both sides of the equation (1.3) and employing Leibniz identity for differentiation give

$$\left(\frac{2t}{e^t + 1} \right)^{(k)} = 2 \left[t \left(\frac{1}{e^t + 1} \right)^{(k)} + k \left(\frac{1}{e^t + 1} \right)^{(k-1)} \right] = \sum_{n=k}^{\infty} G_n \frac{t^{n-k}}{(n-k)!}.$$

In [7, Theorem 2.1] and [17, Theorem 3.1], it was obtained that, when $\lambda > 0$ and $t \neq -\frac{\ln \lambda}{\alpha}$ or when $\lambda < 0$ and $t \in \mathbb{R}$,

$$\left(\frac{1}{\lambda e^{\alpha t} - 1} \right)^{(k)} = (-1)^k \alpha^k \sum_{m=1}^{k+1} (m-1)! S(k+1, m) \left(\frac{1}{\lambda e^{\alpha t} - 1} \right)^m. \quad (2.1)$$

Specially, when $\lambda = -1$ and $\alpha = 1$, the identity (2.1) becomes

$$\left(\frac{1}{e^t + 1} \right)^{(k)} = (-1)^{k+1} \sum_{m=1}^{k+1} (-1)^m (m-1)! S(k+1, m) \left(\frac{1}{e^t + 1} \right)^m. \quad (2.2)$$

Consequently, it follows that

$$\begin{aligned} G_k &= \lim_{t \rightarrow 0} \sum_{n=k}^{\infty} G_n \frac{t^{n-k}}{(n-k)!} = 2k \lim_{t \rightarrow 0} \left(\frac{1}{e^t + 1} \right)^{(k-1)} \\ &= 2k(-1)^k \sum_{m=1}^k (-1)^m (m-1)! S(k, m) \lim_{t \rightarrow 0} \left(\frac{1}{e^t + 1} \right)^m \\ &= (-1)^k k \sum_{m=1}^k (-1)^m \frac{(m-1)!}{2^{m-1}} S(k, m). \end{aligned}$$

The proof of Theorem 1.1 is complete.

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